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A Simplified Solution Using Izbash's Equation for Non-Darcian Flow in a Constant Rate Pumping Test

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Abstract

This paper derives an equivalent of Darcian Theis solution for non-Darcian flow induced by constant rate pumping of a well in a confined aquifer. The derivation, which is valid at later times only, is original. It utilizes Izbash’s equation. This introduces an additional parameter to Darcian condition, namely, empirical exponent. The solution is a non-Darcian equivalent of Jacob straight line method for analyzing pumping tests at late times. It can be used to determine aquifer parameters: storativity, analogous hydraulic conductivity, and empirical exponent. However, while the Jacob method requires a minimum of only one pumping test with one observation well, the additional parameter in the present solution means that a minimum of two observation wells in one test or two pumping tests at different rates with one observation well are required. The derived solution is applied to a case study at Plomeur in Brittany, France, and is shown to provide a practical and efficient method for analyzing pumping tests where non-Darcian groundwater flow occurs.

Introduction

Groundwater flow mostly obeys Darcy’s law with the Reynolds numbers ranging from 1 to 10 (Bear 1972). Outside of this range the flow becomes non-Darcian with a nonlinear relationship of specific discharge and hydraulic gradient (e.g., Bordier and Zimmer 2000; Sen 2000; Wu 2001, 2002a, 2002b; Moutsopoulos and Tsihrintzis 2005). The non-Darcian flow can occur in low permeability materials under very low gradients and when the flow is large through very high permeability media (Hansbo 2001; Xu et al. 2007; Sedghi-Asl et al. 2014; Yang et al. 2017).

For the case of large flow with high permeability, a number of equations have been proposed to quantify the nonlinear relationship between the specific discharge and hydraulic gradient. The most commonly methods were the Forchheimer’s equation (Forchheimer 1901) (a second-order polynomial function) and the Izbash’s equation (Izbash 1931) (a power function). The choice between the two equations is depended on field conditions (Bordier and Zimmer 2000; Yamada et al. 2005; Wen et al. 2006; Yeh and Chang 2013; Houben 2015; Meng et al. 2015). Based on the two equations, many analytical and numerical models for the non-Darcian flow induced by constant rate pumping in different aquifer systems have been derived (e.g., Sen 1990; Wen et al. 2008a, 2008b, 2008c; Wen et al. 2011; Wen et al. 2013; Eck...
An important practical application of the analytical solutions to groundwater flow is in the estimation of aquifer hydraulic properties using drawdown data. For non-Darcian flow, Sen (1989) proposed a curve matching method based on the Boltzmann’ solution. However, determining the most suitable matching is subjective and inefficient due to a complicated computation procedure. Le Borgne et al. (2004) developed other fitting method based on the Barker’s model (Barker 1988). It was successfully used to estimate the hydraulic properties of the Ploemeur aquifer, France. Liu et al. (2016) proposed a generalized non-Darcian radial flow model by which the hydraulic properties could be estimated via a curve-fitting procedure with the least squares methods. Analytical solutions for groundwater flow at greater times can often be approximated by simpler expressions. This makes it easier to apply them to analyze pumping tests, especially in field conditions. Analytical solutions for late time drawdown are also of interest because pumping test data can become more accurate with time as conditions in a pumping test stabilize (Wen et al. 2008b; Liu et al. 2016).

A literature review of published non-Darcian flow studies suggests that there is a lack of research on the analytical solution on the late-time drawdown for non-Darcian flow in pumping tests in confined aquifers. Therefore, a new simplified analytical solution is derived here for this situation. The derivation utilizes Izbash’s equation and the Boltzmann transform. The solution is a non-Darcian equivalent of the Jacob straight line method for analyzing pumping tests at later times. Apart from aquifer storativity and an analogous hydraulic conductivity, the solution contains a third aquifer parameter, the empirical exponent from Izbash’s equation. All three parameters can be determined by applying the solution in pumping tests. The effectiveness of the solution in practice is evaluated using data from pumping tests performed at the Ploemeur site in Brittany, France.

**Problem Statement**

To derive the analytical solution, the same assumptions as those for the Theis equation in the case of Darcian flow are made as: (1) the aquifer is confined, homogeneous, and horizontally isotropic; (2) the aquifer has constant thickness, and is infinite in the horizontal direction; (3) the pumping well and the observation wells fully penetrate the aquifer; (4) the pumping rate is constant over time; and (5) the aquifer is hydrostatic before pumping. A schematic diagram of the aquifer is shown in Figure 1.

According to the mass conservation, the governing equation for two-dimensional radial flow during the pumping period (e.g., Wen et al. 2008b) is

$$\frac{\partial q}{\partial r} + \frac{q}{r} = \frac{S}{b} \frac{\partial s}{\partial t}$$  \hspace{1cm} (1)

where \(b\) is the aquifer thickness [L], \(r\) is the radial distance from the pumping well [L], \(t\) is the pumping time [T], \(q(r, t)\) is specific discharge [LT\(^{-1}\)], \(s(r, t)\) is the drawdown [L], and \(S\) is the dimensionless storativity.

The far-field boundary condition is

$$s(r \to \infty, t) = 0.$$  \hspace{1cm} (2)

The boundary condition representing the fully penetrating well is

$$\lim_{r \to 0} 2\pi b r q = -Q$$  \hspace{1cm} (3)

where \(Q\) is the pumping rate [L\(^3\)T\(^{-1}\)].

The initial condition is

$$s(r, 0) = 0$$  \hspace{1cm} (4)

For modeling purposes for the non-Darcian flow, Basak (1977) and Bordier and Zimmer (2000) pointed out that the Izbash’s equation is likely to be preferred, because it is comparable with the Darcy’s law. Hence, we will employ the Izbash’s equation in the study. Izbash’s equation for the specific discharge is

$$q = \left( \frac{K}{r b} \right)^{\frac{1}{2n}}$$  \hspace{1cm} (5)

where \(n\) and \(K\) are empirical constants. The \(n\) constant is a dimensionless constant of the non-Darcian flow within the range 1 to 2, representing the degree of deviation from linearity. The coefficient, \(K\), is analogous to hydraulic conductivity with the unit as [LT\(^{-1}\)]. When \(n\) value equals to one, the pumping flow becomes Darcian and \(K\) is hydraulic conductivity.

**Linearization Method**

Based on Equation 5, the derivative of \(q\) with respect to \(r\) is

$$\frac{\partial q}{\partial r} = \frac{1}{n} K^{\frac{1}{2n}} \left( \frac{\partial s}{\partial r} \right) \frac{1}{n} \frac{\partial^2 s}{\partial r^2}.$$  \hspace{1cm} (6)

Substituting Equation 6 into Equation 1 yields

$$\frac{\partial^2 s}{\partial r^2} + \frac{n}{r} \frac{\partial s}{\partial r} = \frac{S}{b} \left( \frac{s}{b} \right)^{\frac{n+1}{n}} \frac{\partial s}{\partial t}$$  \hspace{1cm} (7)

To linearize such Equation 7, an approximation is given as

$$\frac{\partial s}{\partial r} = \frac{(q)^n}{K} \approx -\frac{Q}{s^{\frac{n}{n+1}}}.$$  \hspace{1cm} (8)

Equation 8 implies that the water amount through any radial face per unit time is approximately equal to \(Q\). The error of this approximation is small at places near to
the pumping well or at late time (e.g., Odeh and Yang 1919; Ikoku and Ramey Jr 1979; Qian et al. 2005; Wen et al. 2008b). After substituting Equation 8 to Equation 7, the linearized governing equation of the non-Darcian flow is obtained as

\[ \frac{\partial^2 s}{\partial r^2} + \frac{n}{r} \frac{\partial s}{\partial r} = \varepsilon r^{1-n} \frac{\partial s}{\partial t} \] (9)

where \( \varepsilon = \frac{n}{K} \left( \frac{Q^2}{2\pi rb} \right)^{n-1} \frac{S}{b} \).

Substituting Equation 5 into Equation 3, the boundary condition representing the fully penetrating well becomes

\[ \lim_{r \to 0} 2\pi br \left( K \frac{\partial s}{\partial r} \right)^{\frac{1}{n}} = -Q. \] (10)

The far-field boundary condition (Equation 2) and initial boundary condition (Equation 4) remain the same.

**Boltzmann Transform Solution**

Employing the Boltzmann transform, the solution of Equation 9 subject to the boundary conditions (2), (4), and (10) gives the drawdown under non-Darcian flow as

\[ s = \frac{Q}{4K\pi b} \left( \frac{Q}{2\pi rb} \right)^{n-1} \int_{\frac{\beta r^2}{2\pi rb}}^{\infty} \frac{\exp\left(-y\right)}{y} \, dy. \] (11)

The derivation process is documented in the Supporting Information and is original. It is necessary to highlight that Equation 11 is only valid near to the well or at late time because it has been derived using Equation 8, which is subject to this condition. Assuming dimensionless items of \( p, v, \) and \( s' \) as

\[ p = \frac{nSr^2}{4Kbt} \left( \frac{Q}{2\pi rb} \right)^{n-1} \] (12a)

\[ v = \left( \frac{Q}{2\pi r K^{\frac{1}{n}} b} \right)^{n-1} \] (12b)

\[ s' = \frac{4K^{\frac{1}{n}} \pi b}{Q}. \] (12c)

Equation 11 can be rewritten as

\[ s' = W(p) = v \int_{p}^{\infty} \frac{\exp\left(-y\right)}{y} \, dy \] (13)

where \( W(p) \) is the well function for the non-Darcian flow. If \( n = 1 \), the value of \( v \) is equal to 1 and, thus, the \( W(p) \) is reduced to \( W\left( \frac{Sr^2}{4Kbt} \right) \), the well function for Darcian flow.

**Approximate Analytical Solution for Late-Time Drawdown**

Similar to the derivation of Jacob’s method for Darcian flow (Kruseman and Ridderna 1991), the well function of the non-Darcian flow (Equation 13) can be written as

\[ W(p) = v \left[ -0.5772 - \ln p + p \left( \frac{p^2}{2.2!} \right) + \frac{(p)^3}{3.3!} - \ldots \right]. \] (14)

According to Equation 12a, the value of \( p \) becomes small if the radius, \( r \), is small or the pumping time, \( t \),
is large. When \( p < 0.01 \), the \( p \) and its higher order terms in Equation 14 can be neglected (Kruseman and Ridderna 1991). Hence, the well function at the late time can be rewritten as

\[
W(p) = v \left[ -0.5772 - \ln p \right].
\]  
(15)

After changing from natural to 10-base logarithm in Equation 15 and subjecting it into Equation 13, the late-time solution is given as

\[
s = \frac{2.3Q}{4\pi K^\frac{1}{2} b} v \log 0.562 \frac{1}{p}. \tag{16}
\]

It is indicated that the late-time drawdown curve can be presented as a straight line for variable of \( \log 0.562 \frac{1}{p} \) with a constant slope of \( \frac{2.3Q}{4\pi K^\frac{1}{2} b} \). Equation 16 is a valid solution for non-Darcian flow but not a solution for small \( t \) and/or big \( r \).

Interpretation of Drawdown for Non–Darcian Flow

The assessments of the hydraulic properties (\( K \), \( S \), and \( n \)) of the pumped aquifer can be satisfied by observations at least two different constant rate pumping tests (Figure 2). Applying similar derivation to the Jacob method as in Kruseman and Ridderna (1991), the ratio of the two slopes is given by using the late-time drawdown solution (Equation 16) as

\[
\left( \frac{Q}{Q'} \right)^n \left( \frac{r}{r'} \right)^{n-1} = \frac{s_2-s_1}{\log{t_2}-\log{t_1}}. \tag{17}
\]

The value of the constant \( n \) can be obtained as the solution of Equation 17. Based on the constant \( n \), the analysis proceeds as in the Jacob method to get \( K \) and \( S \) as

\[
K = \frac{2.3Q^n (\log{t_2} - \log{t_1})}{4\pi b^n (s_2 - s_1)} \left( \frac{1}{2\pi r} \right)^{n-1}. \tag{18}
\]

\[
S = \frac{K^\frac{1}{2} 2.25bt_0}{v} \frac{1}{nr^2}. \tag{19}
\]

Case Study

The Ploemeur aquifer is the main source of drinking water in the south coast of Brittany, France. To characterize the Ploemeur aquifer after groundwater exploration since 1990, two constant rate pumping tests were carried out as documented by Le Borgne et al. (2004) and Liu et al. (2016). They were a long-term test (LT) in June 1991 and a short-term test (ST) in September 1995. All wells fully penetrated the aquifer. The former studies (Le Borgne et al. 2004; Liu et al. 2016) indicated that flow in the Ploemeur aquifer was non-Darcian. Hence, the data from these tests can be used to evaluate the analytical solution developed in the paper.

To meet the requirement of the use of the developed late-time solution, that is, \( p < 0.01 \), the drawdown from the observation well with small radial distance or from the sufficiently LT is preferred. As a result, the drawdown data from the STs with the radial distances of 46 m and 61 m are chosen to determine the hydraulic properties of the Ploemeur aquifer and the LT with a radial distance of 330 m is adopted to verify the proposed solution for drawdown simulation of the non-Darcian flow (Figure 4a and 4b) in the case. The variable values associated with the pumping tests are given in Table 1. Accordingly, the locations of the pumping well and the observation wells for the pumping tests used in the case study are given in Figure 3.

With the drawdown-time curve, the procedures for parameter assessments are highlighted as following. Firstly, the slopes of the late-time drawdown curves from the STs with the radial distances of 46 m and 61 m can be directly obtained in Figure 4a as 1.15 and 1.1. Secondly, the \( n \) value is estimated as 1.13 by using Equation 17, indicating a moderate non-Darcian flow. And then, based on the \( n \) value, the analogous hydraulic conductivity coefficient, \( K \), is obtained as \( 2.08 \times 10^{-6} \) [m/s] by using Equation 18. Finally, the \( S \) value can be obtained by using Equation 19 as \( 3.03 \times 10^{-3} \).

A comparison between our results of parameter estimation and those from the several previous studies is
shown in Table 2. It is suggested the $K$ values are similar for the three results with $n = 1.13$, and for the two results with $n = 1$. Similarly, the $S$ values seem remarkably consistent in all cases. Assuming that the Ploemeur aquifer is under the Darcian condition, the results from the Theis solution are also given in Table 2. It is indicated that the $K$ and $S$ values of the aquifer under the Darcian condition are greater than that under the non-Darcian condition.

With the estimated hydraulic properties, the drawdown-time curve of the $LT$ can be simulated by our analytical solution of Equation 11 and the former studies of Sen (1990) and Wen et al. (2008b) via software, namely, MATLAB. It is clear to see that the drawdown curve by the proposed solution is well matched with that from the field work and the former studies (Figure 5) at late time. Otherwise, a good match at most data suggests that the error due to larger $p$ is not very significant in this case, possibly because the $n$ value is not very much greater than 1. It proves that conditions in the tests at Plomeur approximate the assumptions of the proposed solution.

### Conclusion

In this paper, a simplified analytical solution using Izbash’s equation is developed for the non-Darcian flow induced by pumping of a well in a confined aquifer. The solution is derived by the Boltzmann transform for late-time drawdown. The result indicates that the late-time

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<tr>
<td>$S$</td>
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<td>$3.10 \times 10^{-3}$</td>
<td>$2.92 \times 10^{-3}$</td>
<td>$2.95 \times 10^{-3}$</td>
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Figure 3. Locations of the pumping well and observation wells of concern in the case study (after Liu et al. 2016).

Figure 4. Drawdown-time series for (a) the STs (b) the $LT$ in the semilog plot (after Liu et al. 2016).

Figure 5. Drawdown simulations for $LT$ with radial distance of 330 m.
drawdown-time curve is approximated as a straight line in the semilog plot.

For data interpretation, the analytical solution can be used for estimation of the power law index, analogous hydraulic conductivity, and storativity of the pumped aquifer by observations at least two different constant rate pumping tests. The case of the constant rate pumping tests performed during 1990–1995 at the Plomeur site in Brittany, France, is studied to investigate the practical application of the proposed model. Being compared with those of previous studies, the estimated hydraulic properties by our analytical solution can well characterize the hydraulic properties in the aquifer conditions on the field scale after the groundwater exploitation, and the drawdown can be well predicted at late time in a straightforward way.

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